Circularly compatible ones and proper circular-arc bigraphs

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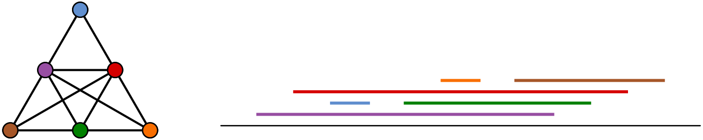
Interval graphs

Interval graphs

A graph is an interval graph if it is possible to assign an interval to each vertex so that two different vertices are adjacent if and only if the corresponding intervals intersect.



The set of intervals is an interval model of the graph.

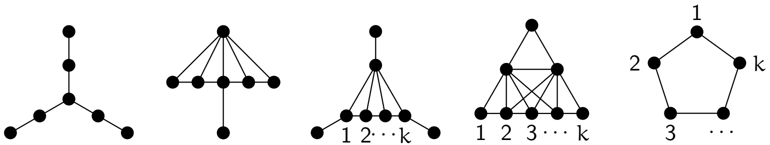
Hajós (1957) posed the problem of characterizing interval graphs.

Since then, several different characterizations have been obtained.

Interval graphs and a certifying recognition algorithm

Theorem (Lekkerkerker and Boland, 1962)

A graph is an interval graph if and only if it does not contain any of the following graphs as induced subgraphs:



bipartite claw | umbrella | k-net, $k \geqslant 2$ | k-tent, $k \geqslant 3$ | C_k, $k \geqslant 4$

Algorithms for recognizing interval graphs and producing an interval model (if the input graph is an interval graph) have long been known; e.g., Fulkerson and Gross (1965) (first polynomial-time algorithm) and Booth and Lueker (1976) (first linear-time algorithm).

Lindzey and McConnell (2016) developed a linear-time algorithm that, given a graph that is **not** an interval graph, finds one of the forbidden subgraphs in the above theorem as an induced subgraph.

Put together give rise to a certifying linear-time recognition algorithm.

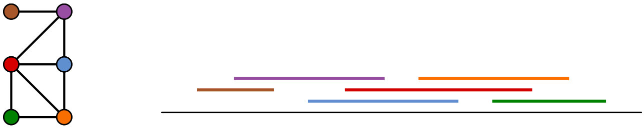
Proper interval graphs

One of the most studied subclasses of interval graphs is the class of proper interval graphs.

Proper interval graphs

A graph is a proper interval graph if it is possible to assign an interval to each vertex so that:

- two vertices are adjacent if and only if the corresponding intervals intersect, and
- no two of these intervals are one properly contained in the other.

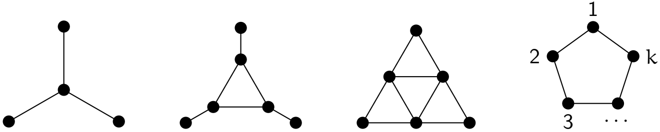


Such a set of intervals is called a proper interval model of the graph.

Proper interval graphs

Theorem (Wegner, 1967; Roberts, 1969)

A graph is a proper interval graph if and only if it does not contain any of the following graphs as induced subgraphs:



claw net tent C_k, $k \geqslant 4$

Theorem (Looges and Olariu, 1993; Corneil et al., 1995; Hell and Huang, 2004)

There is a linear-time algorithm that, given any graph, decides whether or not it is a proper interval graph. Moreover, in the affirmative case, it returns a proper interval model and, in the negative case, a minimal forbidden induced subgraph.

Among many other characterizations of proper interval graphs, there is a matrix characterization due to Roberts.

Linearly compatible ones property

In connection with Roberts' matrix characterization of proper interval graphs, Tucker (1969) introduced the linearly compatible ones property.

We say a row of a matrix is trivial if it has only 0's or only 1's.

Linearly compatible ones property

A matrix has the linearly compatible ones property if there is a permutation of its rows and columns so that:

- the 1's in each row form an interval in such a way that, excluding the trivial rows, the sequences of left and of right endpoints of these intervals are non-decreasing, and
- the 1's in each column also form an interval.

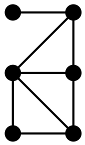
$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proper interval graphs and linearly compatible ones

An augmented adjacency matrix is obtained from an adjacency matrix by adding 1's all along the main diagonal.

Theorem (Roberts, 1971)

A graph is a proper interval graph if and only if its augmented adjacency matrix has the linearly compatible ones property.



$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

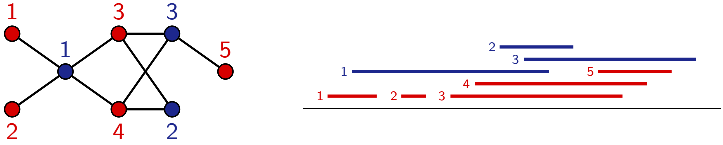
Proper interval bigraphs

A graph is bipartite if its vertices can be colored red and blue so that adjacent vertices have different colors. The partition of the vertex set into the sets of red and blue vertices is called a bipartition.

Proper interval bigraphs (Sen and Sanyal, 1994)

A proper interval bigraph is a bipartite graph such that it is possible to assign an interval to each vertex so that:

- vertices on different sides of the bipartition are adjacent if and only if the corresponding intervals intersect, and
- there are no two vertices on the same side of the bipartition whose intervals are one properly contained in the other.

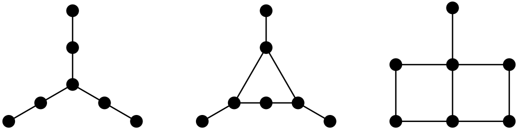


Such a set of intervals is called a proper interval bimodel of the graph.

Proper interval bigraphs

Theorem (Steiner, 1996)

A bipartite graph is a proper interval bigraph if and only if it contains as induced subgraphs neither C_{2k} for any $k \geqslant 3$ nor any of the following:



Theorem (Spinrad, Brandstädt, and Stewart, 1987; Hell and Huang, 2004)

There is a linear-time algorithm that, given any bipartite graph, decides whether or not it is a proper interval bigraph. Moreover, in the affirmative case it returns a proper interval bimodel and, in the negative case, it returns a minimal forbidden induced subgraph.

D-interval property

There is also a matrix characterization of proper interval bigraphs in terms of the D-interval property.

D-interval property (Moore, 1977)

A matrix has the D-interval property if it is possible to permute its columns so that:

- the 1's in each row form an interval, and
- if two such intervals are one contained in the other, then they share an endpoint.

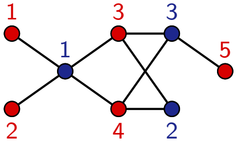
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \underline{1} & \underline{1} & 0 & 0 \\ 0 & \underline{1} & \underline{1} & \underline{1} & 0 \\ \underline{1} & \underline{1} & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & \underline{1} & 0 \end{pmatrix}$$

Proper interval bigraphs and the D-interval property

Bipartite graph associated with a matrix. Biadjacency matrix.

The bipartite graph G associated with a matrix $M = (m_{ij})$ has a vertex for each row and for each column and the edges are precisely those joining the vertex of row i with the vertex of column j when $m_{ij} = 1$. Moreover, the matrix M is called a biadjacency matrix of G.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \underline{1} & \underline{1} & \underline{1} & \underline{1} & 0 \\ 0 & 0 & \underline{1} & \underline{1} & 0 \\ 0 & 0 & \underline{1} & \underline{1} & \underline{1} \end{pmatrix} \end{array} \longrightarrow$$



Theorem (Steiner, 1996)

Proper interval bigraphs are precisely the bipartite graphs associated with matrices having the D-interval property.

Proper interval bigraphs and linearly compatible ones

Thus we have two kinds of characterizations:

- proper interval graphs by the linearly compatible ones property
- proper interval bigraphs by the D-interval property

Relying on results by Moore (1977) and Lai and Wei (1997), we show:

Theorem (S., 2021)

The linearly compatible ones property coincides with D-interval property.

This means that both classes can characterized in terms of the linearly compatible ones property only.

Graph classes with the linearly compatible ones property

	...has the linearly compatible ones property
the augmented adjacency matrix...	**proper interval graphs**[1]
the biadjacency matrix...	**proper interval bigraphs**[2]

[1]Roberts (1971)
[2]Steiner (1996) and S. (2021)

Proper circular-arc graphs

Proper circular-arc graphs (Tucker, 1969)

A graph is a proper circular-arc graph if it is possible to assign an arc of a circle to each vertex so that:

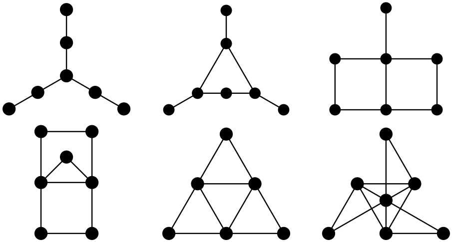
- two vertices are adjacent if and only if the corresponding arcs intersect, and
- no two such arcs are one properly contained in the other.

Such a set of arcs is called a proper circular-arc model of the graph.

Proper circular-arc graphs

Theorem (Tucker, 1974)

A graph is a proper circular-arc graph if and only if does not contain as an induced subgraph $C_k \cup K_1$ ($k \geqslant 4$), $\overline{C_{2k}}$ ($k \geqslant 3$), $\overline{C_{2k+1} \cup K_1}$ ($k \geqslant 1$) nor the complement of any of the following graphs:



Theorem (Deng, Hell, and Huang, 1996; Nussbaum, 2008)

There is a linear-time algorithm that, given any graph, decides whether or not it is a proper circular-arc graph. Moreover, in the affirmative case it returns a proper circular-arc model and, in the negative case, it returns a minimal forbidden induced subgraph.

Circularly compatible ones property

Tucker (1969) introduced the circularly compatible ones property to achieve for proper circular-arc graphs a characterization analogous to that of Roberts for proper interval graphs.

Circularly compatible ones property

A matrix has the circularly compatible ones property if there is a permutation of its rows and columns so that:

- the 1's in each row form a circular interval in such a way that, excluding the trivial rows, the sequences of left and of right endpoints of these circular intervals are circularly monotone, and
- the 1's in each column form a circular interval.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Such a permutation of rows and columns is called a circularly compatible ones biorder of the matrix.

Proper circular-arc graphs and circularly compatible ones

Recall:

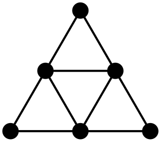
Theorem (Roberts, 1971)

A graph is a proper interval graph if and only if its augmented adjacency matrix has the linearly compatible ones property.

Tucker proved an exact analogue for proper circular-arc graphs:

Theorem (Tucker, 1969)

A graph is a proper circular-arc graph if and only if its augmented adjacency matrix has the circularly compatible ones property.



$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Two problems by Tucker

Problems (Tucker, 1969)

For the circularly compatible ones property in arbitrary matrices (not restricted to augmented adjacency matrices):

1. find a forbidden submatrix characterization and
2. find an efficient recognition algorithm.

One of our main results is the solution to these problems.

Circularly compatible ones property

Theorem (S., 2021)

A matrix has the circularly compatible ones property if and only if it does not contain as a submatrix any of the following matrices or their transposes, or permutations of rows and columns of them:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\quad
\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 1 & & & & 0 \\ & 1 & 1 & & & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}
\quad
\begin{pmatrix} 0 & 0 & & & & 1 \\ & 0 & 0 & & & 1 \\ & & \ddots & \ddots & & \vdots \\ & & & 0 & 0 & 1 \\ 0 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix}
$$

Theorem (S., 2021)

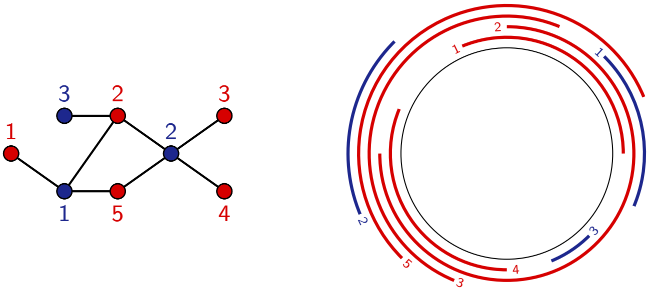
There exists a linear-time algorithm that, given any matrix, decides whether or not it has the circularly compatible ones property. Moreover, in the affirmative case it returns a circularly compatible ones biorder and, in the negative case, a minimal forbidden submatrix.

Proper circular-arc bigraphs

Proper circular-arc bigraphs (Hell and Huang, 2004)

A proper circular-arc bigraph is a bipartite graph such that it is possible to assign an arc of a circle to each vertex so that:

- vertices on different sides of the bipartition are adjacent if and only if the corresponding arcs intersect, and
- there are no two vertices on the same side of the bipartition whose arcs are one properly contained in the other.



If so, the set of arcs is called a proper circular-arc bimodel of the graph.

Proper circular-arc bigraphs: characterization

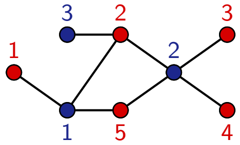
Basu, Das, Ghosh, and Sen (2013) characterized the subclass of proper circular-arc bigraphs having biadjacency matrices without trivial rows.

We characterized the whole class of the proper circular-arc bigraphs.

Theorem (S., 2021)

The proper circular-arc bigraphs are precisely the bipartite graphs associated with the matrices having the circularly compatible ones property.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \longrightarrow$$



Graph classes with the compatible ones properties

	...has the linearly compatible ones property	...has the circularly compatible ones property
the augmented adjacency matrix...	**proper interval graphs**[1]	**proper circular-arc graphs**[2]
the biadjacency matrix...	**proper interval bigraphs**[3]	**proper circular-arc bigraphs**[4]

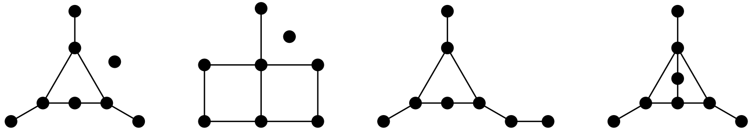
[1] Roberts (1971)
[2] Tucker (1969)
[3] Steiner (1996) and S. (2021)
[4] S. (2021)

Proper circular-arc bigraphs: structure

By virtue of the equivalence between proper circular-arc bigraphs and the circularly compatible ones property, our characterization of the latter in terms of forbidden submatrices implies the following.

Theorem (S., 2021)

A bipartite graph is a proper circular-arc bigraph if and only if it contains as induced subgraphs neither $C_{2k} \cup K_1$ nor its bipartite complement for any $k \geqslant 3$, nor any of the following graphs:



Proper circular-arc bigraphs: recognition

Basu, Das, Ghosh, and Sen (2013) posed the problem of finding an efficient algorithm for recognizing proper circular-arc bigraphs. The same problem was reproposed by Das and Chakraborty (2017).

Thanks to the equivalence between proper circular-arc bigraphs and the circularly compatible ones property, together with our recognition algorithm for the latter, we also obtain a solution to this problem.

Theorem (S., 2021)

There exists a linear-time algorithm that, given any bipartite graph, decides whether or not it is a proper circular-arc bigraph. Moreover, in the affirmative case it returns a proper circular-arc bimodel and, in the negative case, a minimal forbidden induced subgraph.

Interestingly, from these results for proper circular-arc bigraphs it is possible to derive the known analogous results for the proper interval bigraphs (Steiner, 1996; Spinrad, Brandstädt, and Stewart, 1987; Hell and Huang, 2004).

D-circular property

In fact, we do not obtain our results for the circularly compatible property directly, but we derive them from more general results for the following variant of the D-interval property.

D-circular property (Köbler, Kuhnert, and Verbitsky, 2016)

A matrix has the D-circular property if there exists some column permutation such that:

- the 1's in each row form a circular interval, and
- if two such circular intervals are one contained in the other, then they share an endpoint.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{1} & \underline{1} & 0 & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} & \underline{1} \end{pmatrix}$$

If so, the column permutation is called a D-circular order of the matrix.

D-circular property and circular-ones property

Remark

A matrix M has the D-circular property iff the matrix $D(M)$ that arises by adding as new rows all the differences $r - s$ for each row r that dominates a row s has the circular-ones property (i.e., admits a column permutation such that in each row the 1's form a circular interval).

$$M \qquad \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\downarrow$$

$$D(M) \qquad \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_1 - r_4 \\ r_2 - r_3 \\ r_2 - r_4 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ \underline{1} & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{1} & 0 \\ 0 & \underline{1} & 0 & 0 & \underline{1} \end{pmatrix} \longrightarrow \begin{pmatrix} \underline{1} & \underline{1} & 0 & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 0 & \underline{1} & \underline{1} & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} & \underline{1} \\ \underline{1} & \underline{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} \\ 0 & \underline{1} & \underline{1} & 0 & 0 \end{pmatrix}$$

D-circular property: structure

Theorem (S., 2021)

A matrix has the D-circular property if and only if it does not have as a submatrix any of the following matrices up to permutations of rows and columns:

$$\begin{pmatrix}1&1&1&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} \quad \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&1&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&0&1&0\end{pmatrix} \quad \begin{pmatrix}1&1&1&0&0\\0&1&1&1&0\\0&0&1&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&0&1\\1&1&0&0\\1&1&1&0\\0&1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&0\\0&1&1&1\\0&0&1&0\\1&0&0&0\end{pmatrix} \quad \begin{pmatrix}1&1&1&0\\1&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix} \quad \begin{pmatrix}1&1&1&0\\1&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix} \quad \begin{pmatrix}1&1&1&0&1\\0&1&1&1&1\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix} \quad \begin{pmatrix}0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\1&0&1&1\end{pmatrix} \quad \begin{pmatrix}0&0&0&1&1\\1&0&0&0&1\\1&1&0&1&1\end{pmatrix} \quad \begin{pmatrix}0&0&0&1\\0&1&1&0\\1&0&1&1\\1&1&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&&&&0\\&1&1&&&0\\&&\ddots&\ddots&&\vdots\\&&&1&1&0\\1&0&\cdots&0&1&0\end{pmatrix} \quad \begin{pmatrix}0&0&&&&1\\&0&0&&&1\\&&\ddots&\ddots&&\vdots\\&&&0&0&1\\0&1&\cdots&1&0&1\end{pmatrix}$$

The proof relies on the preceding remark and on a characterization by forbidden submatrices for the circular-ones property from an earlier work (S., 2020).

D-circular property: recognition

Theorem (S., 2021)

There exists a linear-time algorithm that, given any matrix, decides whether or not it has the D-circular property. Moreover, if yes, it returns a D-circular order and, if no, it returns a minimal forbidden submatrix.

- The result does not follow by applying to $\mathrm{D}(M)$ a linear-time algorithm for recognizing the circular-ones property because the size of $\mathrm{D}(M)$ may be quadratic in the size of M.
- We introduce a matrix $\Delta(M)$, whose size in linear in the size of M if M has the D-circular property, and prove that M has the D-circular property if and only if $\Delta(M)$ has the circular-ones property.
- Moreover, the structural result on the previous slide implies that the D-circularity of M is equivalent to M having the circular-ones property plus M not containing one of the 13 sporadic matrices.
- The algorithm follows by relying on a certifying linear-time recognition algorithm for the circular-ones property (S., 2020) and adapting an algorithmic technique by Lindzey and McConnell (2016) for detecting sporadic submatrices for the consecutive-ones property.

How is D-circularity related to circularly compatible ones?

Theorem (S., 2021)

For each matrix M, the following assertions are equivalent:

- M has the circularly compatible ones property;
- both M and its transpose have the D-circular property.

Hence, the structural and algorithmic results for the D-circular property translate into the analogous results the circularly compatible ones property we have presented.

Open problems

1. Characterize the whole class of circular-arc graphs by forbidden structures that can be found in less than $O(n^3)$ time.

   The $O(n^3)$-time bound is matched by an algorithm by Francis, Hell, and Stacho (2015).

2. Characterize the class of interval bigraphs by forbidden structures.

   Partial results were obtained by Das, Das, and Sen (2016).

3. Characterize the class of unit circular-arc bigraphs by forbidden structures.

   This class is obtained by requiring the arcs to be closed and all of the same same length.

   It is known that it is different from the class of proper circular-arc bigraphs (Basu, Das, Ghosh, and Sen, 2013).

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Thank you very much for your attention!