Lochs' index: weight function and change of basis

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Joint work with
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Motivation: simulating continued fractions

For computer simulation:

- Given t binary digits $b_1, b_2, \ldots, b_t$ of $x \in [0,1]$,

$$x = (0.b_1 b_2 \ldots)_2 \in [0,1]\,.$$

- Number $n = n_t(x)$ of CFE-digits (partial quotients) deduced without possibility of error ?

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}\,.$$

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Natural to consider the quotient $n_t(x)/t$:

- rate of CFE digits per binary digit,
- compares relative information/redundancy of expansions.

First historical results: Lochs' Theorem

Theorem (Lochs '64)

The rate of CF-digits per decimal given satisfies

$$\lim_{\mathtt{d}\to\infty} \frac{n_{\mathtt{d}}(x)}{\mathtt{d}} = \frac{6\log 2 \log 10}{\pi^2} \doteq 0.9702701\ldots,$$

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Theorem (Faivre '98)

$$\Pr\left\{x\in[0,1] : \frac{n_{\mathtt{d}}(x) - \mathtt{d}\times a}{\sigma\sqrt{\mathtt{d}}} \le \theta\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\theta} e^{-u^2/2}du\,,$$

where $a = \frac{6\log 2\log 10}{\pi^2}$ is the Lochs' constant, and $\sigma > 0$.

Change of basis: a simple example

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Answer:

- For $d = 2^A$ we simply obtain
$$A \times n_t(x) = t ,$$
because 1 d-ary digit corresponds to A binary digits.

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- More generally we expect
$$\frac{\log d}{\log 2} \times n_t(x) \sim t \,,$$
one digit in base d^B corresponds to one in base 2^A when $d^B \approx 2^A$.

Motivation: source transformation

Classical

- **Dajani&Fieldsteel'01**: From source S_1 to S_2, both of positive entropy:

$$\lim L_t(x; S_1, S_2)/t = h(S_1)/h(S_2)\,,$$

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  - If $h(S_2) = 0$ and $h(S_1) > 0$, almost surely $L/t \to \infty$.
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Our work

- Introduce appropriate notion of *renormalized* entropy f_1, f_2,
- Generalization: for positive, zero or infinite entropy:

$$\lim f_2(L_t(x; S_1, S_2))/f_1(t) = 1\,.$$

Plan of the talk

1. Definitions: partitions, Lochs' and weight function
2. Statement of main result and discussion
3. Examples of natural zero entropy sources that have weight
4. Concepts for the proof of the main result
5. Conclusions

Section

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Intervals: sources and partitions

Definition (System of interval partitions)

Sequence of topological partitions $\mathcal{P} = (\mathcal{P}_n)$ of $[0,1]$

- $\mathcal{P}_{n+1}$ refinement of $\mathcal{P}_n$ for every n.
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Equivalent to sources

- notation $I_n^{\mathcal{P}}(x) = I \in \mathcal{P}_n$ such that $x \in I$,
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Example. Decimal expansion

Depth n interval for $x = (0.d_1 d_2 \ldots)_{10}$

$$I_n^{\mathcal{D}}(x) = \left((0.d_1 \ldots d_n)_{10}, (0.d_1 \ldots d_n)_{10} + 10^{-n}\right) ,$$

containing $y \in (0, 1)$ having the exact same first n digits as x.

Entropy of a partition

Entropy dictates size of intervals

- *Shannon entropy*[1]:

$$H(\mathcal{P}) = -\lim_{k\to\infty} \frac{1}{k} \sum_{I\in\mathcal{P}_k} |I| \log |I| \ .$$

- *Point-wise*: for almost every x

$$h(\mathcal{P}) = -\lim_{k\to\infty} \frac{1}{k} \log \left| I_k^{\mathcal{P}}(x) \right| \ .$$

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Point-wise to Shannon

$$H(\mathcal{P}) = \lim_{k\to\infty} \mathbb{E}\Big[-\frac{1}{k} \log \left| I_k^{\mathcal{P}}(x) \right| \Big] .$$

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Lochs' index for systems of partitions $\mathcal{P}^1, \mathcal{P}^2$

$$L_n(x; \mathcal{P}^1, \mathcal{P}^2) := \max\{m \geq 0 : I_n^{\mathcal{P}^1}(x) \subset I_m^{\mathcal{P}^2}(x)\},$$

depth in $\mathcal{P}^2$ deduced from depth n in $\mathcal{P}^1$.

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Explanation

If $I_n^{\mathcal{P}^1}(x)$ splits over (intersects) several $J \in \mathcal{P}_m^2$,

$\implies$ we cannot yet decide on $I_m^{\mathcal{P}^2}(x)$

Theorem (Dajani, Fieldsteel, 2001)

Consider systems of partitions $\mathcal{P}^1$ *and* $\mathcal{P}^2$*, with positive point-wise entropies* $h(\mathcal{P}^1)$ *and* $h(\mathcal{P}^2)$*. Then*

$$\lim_{n\to\infty} \frac{1}{n} L_n(x; \mathcal{P}^1, \mathcal{P}^2) = \frac{h(\mathcal{P}^1)}{h(\mathcal{P}^2)}$$

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$$h(\mathcal{C}) = 2 \lim_{k\to\infty} \frac{1}{k} \log q_k(x) = \frac{\pi^2}{6 \log 2}\,.$$

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$\Longrightarrow$ we deduce Lochs' Theorem and the result for d-ary basis.

Existence of point-wise entropy

Systems of partitions associated with good (positive entropy) dynamical systems have point-wise entropy:

Theorem (Shannon,McMillan,Breiman)

Let T be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{B}, \mu)$ and let P be a finite or countable generating partition for T for which $H_\mu(P) < \infty$. Then for μ-a.e. x,

$$\lim_{n\to\infty} -\frac{\log \mu\left(P_n(x)\right)}{n} = h_\mu(T)\,.$$

Here $H_\mu(P)$ denotes the entropy of the partition P, $h_\mu(T)$ the entropy of T and $P_n(x)$ denotes the element of the partition $\bigvee_{i=0}^{n-1} T^{-i}P$ containing x.

Log-balancedness and weight function

Definition (Weight function)

A system of partitions $\mathcal{P} = (\mathcal{P}_n)$ is *log-balanced* a.e. (resp. in measure) with *weight function* $f\colon \mathbb{N} \to \mathbb{R}_{>0}$, $f(n) \to \infty$, if

$$-\log |I_n^{\mathcal{P}}(x)| \sim f(n)\,,$$

almost everywhere (resp. in measure).

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Example

- For positive entropy $h = h(\mathcal{P}) > 0$

$$f(n) = h \times n\,.$$

- If partition is log-balanced, entropy 0 corresponds to

$$f(n) = o(n)\,.$$

Realization result for weight functions

Proposition

Let $f\colon \mathbb{N} \to \mathbb{R}_{>0}$ non-decreasing, $f(n) \to \infty$. Then there exists a log-balanced $\mathcal{P}$ with weight function f almost everywhere.

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Proof sketch.

Given n, let $k = k(n)$ be such that $2^k \leq \exp(f(n)) < 2^{k+1}$.

Define

$$\mathcal{P}_n := \left\{ \left(\frac{i}{2^k}, \frac{i+1}{2^k} \right) : 0 \leq i < 2^k \right\},$$

so that $|I_n(x)| = 2^{-k}$ satisfies $e^{-f(n)} \leq 2^{-k} < 2e^{-f(n)}$. □

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Our main result

Theorem (Berthé,Cesaratto,R.,Safe, 2021+)

Consider systems of partitions $\mathcal{P}^1$ *and* $\mathcal{P}^2$*, with a.e. weight functions* f_1 *and* f_2*. Then, under certain technical conditions*

$$\lim_{n\to\infty} \frac{f_2\left(L_n(x;\mathcal{P}^1,\mathcal{P}^2)\right)}{f_1(n)} = 1\,,$$

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The conditions are:

- $\sum e^{-\delta f_1(n)} < \infty$ for every $\delta > 0$;
- f_2 is non decreasing ;
- $f_2(n+1) - f_2(n) = o(f_2(n))$ as $n \to \infty$.

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Remark. First condition can be dropped for convergence in measure.

Discussion: conditions of our main result

We recall the conditions:

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Intuitively, the first condition is the most constraining one:

- Condition (b) reflects the fact that $\mathcal{P}_2$ is refining ;
- Condition (c) means that $f_2(n+1) \sim f_2(n)$;
- Condition (a) tells us that $f_1(n)$ grows *not too slowly*

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Important remarks

– Condition (a) not satisfied when $f_1(n) = \log n$,
– Condition (a) satisfied for $f_1(n) \geq (\log n)^2$.

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– Condition (c) not satisfied when $f_2(n) = \exp(n)$,
– Condition (c) is satisfied when $f_2(n) = \exp(\sqrt{n})$.

Discussion: conditions of our main result

Example: appropriate output partitions $\mathcal{P}_2$

Subexponential weight functions of the form

$$f_2(n) = \exp(g(n))\,,$$

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Note. For convergence in measure the conditions on the input partitions can be dropped.

Section

Two natural zero entropy sources with weight

Farey partition (Sturm source) and Stern-Brocot partition built by splitting intervals at *mediant*

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Farey partition $\mathcal{F}_n$:

- Base case: $\mathcal{F}_0 = \{[0,1]\}$.
- Building $\mathcal{F}_n$:
  split $\left[\frac{a}{b}, \frac{c}{d}\right] \in \mathcal{F}_{n-1}$,
  if $b + d \leq n + 1$.

Stern-Brocot partition $\mathcal{SB}_n$:

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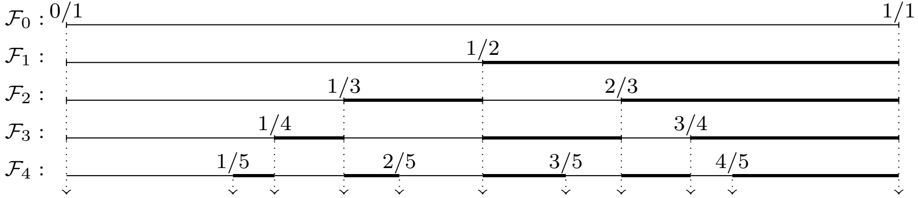

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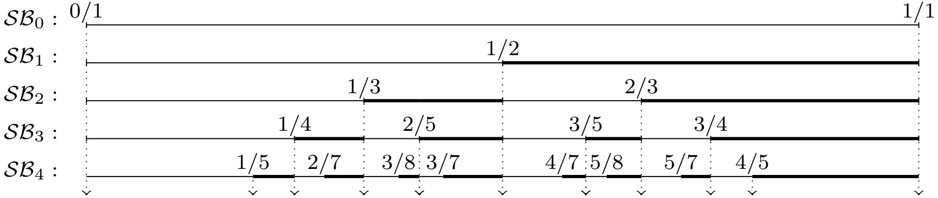

Farey partition

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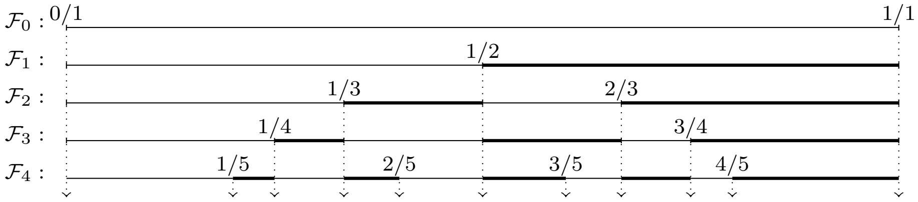

Properties:

- $\mathcal{F}_k$ determines char. Sturmian word up to $(k-1)$-th symbol.
- The end-points $\mathcal{F}_k$ are exactly $\{\frac{a}{b} \in \mathbb{Q} : 0 \leq a \leq b \leq k+1\}$.
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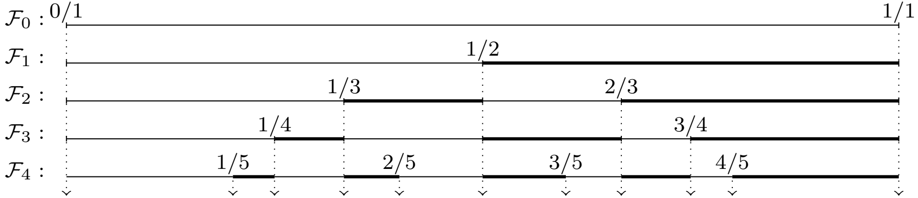

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Weight of the Farey partition

Proposition

Farey partition is log-balanced a.e. with weight-function $f(n) = 2 \log n$.

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Farey intervals have comparable size almost everywhere:

Lemma

For almost every x, for large $n \geq n_0(x)$

$$\frac{1}{n^2} \leq \left|I_n^{\mathcal{F}}(x)\right| \leq \frac{(\log n)(\log\log n)}{n^2}$$

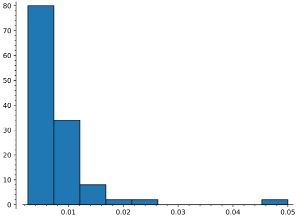
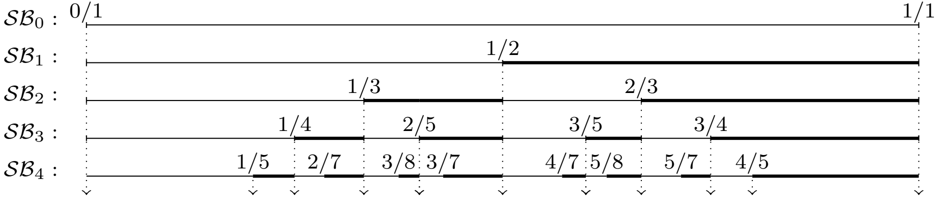


Figure. Histogram of interval sizes for $n = 20$. $\frac{1}{20^2} = 0.0025$, $\frac{1}{20} = 0.05$.

Stern-Brocot partition

Stern-Brocot partition $\mathcal{SB}_n$:

- Base case: $\mathcal{SB}_0 = \{[0,1]\}$.
- Building $\mathcal{SB}_n$: split $\left[\frac{a}{b}, \frac{c}{d}\right] \in \mathcal{SB}_{n-1}$ always.



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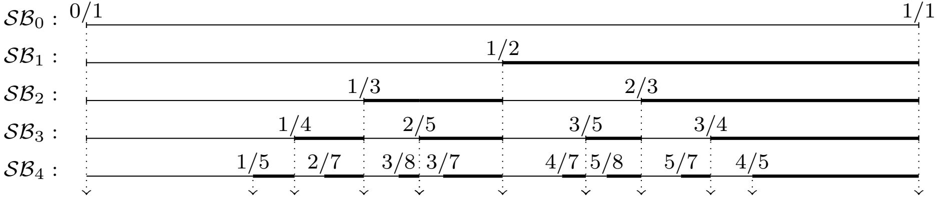

Associated to binary encoding of continued fractions:

$$[a_1, a_2, \ldots] \mapsto [0^{a_1-1}, 1, 0^{a_2-1}, 1, \ldots]$$

which follows construction of CFs by mediants.

Weight of the Stern-Brocot partition

Proposition

Stern-Brocot is log-balanced in measure with weight-function

$$f_{\mathcal{SB}}(n) = \frac{\pi^2}{6} \frac{n}{\log n} \,.$$

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Stern-Brocot system of partitions is not log-balanced almost everywhere.

Proof sketch.

– Depth in Stern-Brocot strongly related to the growth to sum of partial quotients $\sum_{k=1}^{m} a_k(x)$.

– Sum behaves well in measure but erratic almost-everywhere. □

Consequences for our sources of zero-entropy

Corollary 1

Let $\mathcal{P}$ with $h(\mathcal{P}) > 0$ and $\mathcal{SB}$ be the Stern-Brocot partition, then

$$L_n(x; \mathcal{P}, \mathcal{SB}) \sim \frac{6\, h(\mathcal{P})}{\pi^2} \times n \log n\,,$$

in measure.

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Proof.

Since $f_{\mathcal{P}}(n) = h(\mathcal{P}) \times n$ and $f_{\mathcal{SB}}(m) = \frac{\pi^2}{6} \frac{m}{\log m}$ in measure,

$$\frac{\pi^2}{6} \frac{L_n(x; \mathcal{P}, \mathcal{SB})}{\log L_n(x; \mathcal{P}, \mathcal{SB})} \sim h(\mathcal{P}) \times n \,.$$

Applying logs shows that $\log L_n(x; \mathcal{P}, \mathcal{SB}) \sim \log n$ too. □

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Corollary 2

Let $\mathcal{P}$ with $h(\mathcal{P}) > 0$ and $\mathcal{F}$ be the Farey partition, then

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Proof.

For the input $f_1(n) = h(\mathcal{P}) \times n$, for the output $f_2(m) = 2\log m$. □

Second order term: continued fractions to Farey

Second order term might be irregular: big variability in L_n

Proposition

The Lochs' index from continued fractions to Farey satisfies

$$2\log L_n(x;\mathcal{CF},\mathcal{F}) = h(\mathcal{CF}) \times n + c\,Z_n(x) \cdot \sqrt{n} + O(1) \;,$$

where $c > 0$ and $Z_n \Rightarrow N(0,1)$.

Recall. $f_{\mathcal{CF}}(n) = h(\mathcal{CF}) \times n$, and $f_{\mathcal{F}}(m) = 2\log m$.

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Proof.

Find specific formula for L_n in this case, then use CLT for $\log q_k(x)$. □

A "non-example": Farey to continued fractions

Recall. $f_1(n) = 2\log n$ not valid weight function a.e. for input $\mathcal{P}^1$.

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For the Farey $\mathcal{F}$ and the Continued Fraction $\mathcal{CF}$ systems of partitions:

$$\lim_{n\to\infty} \frac{L_n(x;\mathcal{F},\mathcal{CF})}{\log n} = \frac{12\log 2}{\pi^2},$$

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- Follows from characterization of L_n for the given sources.
- Main Theorem only gives this limit in measure

Section

1. Definitions: partitions, Lochs' and weight function

2. Statement of main result and discussion

3. Examples of natural zero entropy sources that have weight

4. Concepts for the proof of the main result

5. Conclusions

Recall: main result

Theorem (Berthé,Cesaratto,R.,Safe, 2021+)

Consider systems of partitions $\mathcal{P}^1$ and $\mathcal{P}^2$, with a.e. weight functions f_1 and f_2. Then, under certain technical conditions

$$\lim_{n\to\infty} \frac{f_2\left(L_n(x;\mathcal{P}^1,\mathcal{P}^2)\right)}{f_1(n)} = 1\,,$$

for a.e. x.

The conditions are:

- $\sum e^{-\delta f_1(n)} < \infty$ for every $\delta > 0$;
- f_2 is non decreasing ;
- $f_2(m+1) - f_2(m) = o(f_2(m))$ as $m \to \infty$.

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$\Rightarrow$ Formal proof separated into two parts: upper-limit and lower-limit.

Proof-sketch: upper-limit

Upper-limit requires almost no conditions at all:

Lemma

Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be a.e. log-balanced with weights f_1 and f_2 respectively. If f_2 is non-decreasing

$$\limsup_{n\to\infty} \frac{f_2(L_n(x;\mathcal{P}^1,\mathcal{P}^2))}{f_1(n)} \leq 1 \qquad a.e.$$

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Proof-sketch for f_2 strictly increasing.

Fix $\varepsilon > 0$. Consider $m > f_2^{-1}\left((1+\varepsilon)\times f_1(n)\right)$, then

$$-\log|I_m^{\mathcal{P}^2}(x)| \sim f_2(m) \geq (1+\varepsilon)\times f_1(n)\,,$$

while $-\log|I_n^{\mathcal{P}^1}(x)| \sim f_1(n)\,.$

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while $-\log|I_n^{\mathcal{P}^1}(x)| \sim f_1(n)$. Thus $I_n^{\mathcal{P}^1}(x)$ too large for $I_m^{\mathcal{P}^2}(x)$. □

Upper-limit: a funny-looking corollary

Corollary

Let $\mathcal{P}$ be a.e. log-balanced with weight f. If f is non-decreasing

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Proof.

We know the upper-limit works. Lower-limit follows from

$$n \leq L_n(x;\mathcal{P},\mathcal{P})\,. \qquad \square$$

Proof-sketch: lower-limit

Lower-limit requires all of the conditions:

Lemma

Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be a.e. log-balanced with weights f_1 and f_2 respectively, satisfying the conditions in the statement of the Theorem, then

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Proof techniques: covering argument + Borel-Cantelli.

Section

Conclusions and further work

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3. Results on average ?

Thank you!

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