Dynamics of Ostrowski's numeration: Limit laws and Hausdorff dimensions

Valérie Berthé, Jungwon Lee

IRIF-CNRS-Université de Paris
http://www.irif.fr/~berthe

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Overview

We study the Ostrowski map defined on $[0,1]^2$ as

$$S(x,y) = (\{1/x\}, \{y/x\})$$

This a simple skew-product extension of the Gauss map

$$T : [0,1] \to [0,1],\; x \mapsto \{1/x\}$$

which produces continued fraction expansions

- We study statistical properties of digits (limit laws for Birkhoff sums).
- We estimate the size of sets defined by bounded digits (Hausdorff dimension).

Dynamically

- Ostrowski's map is defined as $S(x, y) = (\{1/x\}, \{y/x\})$.
- For $(x, y) \in [0, 1]^2$, set

$$(x_0, y_0) := (x, y), \quad (x_i, y_i) := S^i(x, y) \quad \text{for all } i \geq 1.$$

- We get a sequence of (pairs of) digits

$$(a_i, b_i) = \left(\left\lfloor \frac{1}{x_{i-1}} \right\rfloor, \left\lfloor \frac{y_{i-1}}{x_{i-1}} \right\rfloor \right).$$

- The sequence $(a_i)_i$ provides the continued fraction expansion of $x = [0; a_1, a_2, \cdots]$.

$$x_1 = 1/x - a_1 \rightsquigarrow x = \frac{1}{a_1 + x_1}$$

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- Set

$$\theta_i := |q_i x - p_i|, \quad \text{with } p_i/q_i = [0; a_1, \cdots, a_i].$$

The sequence $(b_i)_i$ yields the digits for the Ostrowski representation of y w.r.t. the irrational base x

$$y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}|$$

Arithmetically

Consider the second coordinate

$$S(x, y) = (\{1/x\}, \{y/x\}) = (1/x - a, y/x - b)$$

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$$x_1 = 1/x - a_1 \rightsquigarrow x = \frac{1}{a_1 + x_1}$$

$$y_1 = y/x - b_1 \rightsquigarrow y = x(b_1 + y_1)$$

$$y_{i-1} = x_{i-1}(b_i + y_i)$$

$$y = \sum_{i=1}^{n} b_i x_0 x_1 \cdots x_{i-1} + x_0 x_1 \cdots x_{n-1} y_n.$$

We then use the identity $x_0 x_1 \cdots x_i = (-1)^i \theta_i = |q_i x - p_i|$

$$\rightsquigarrow y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}|$$

On Ostrowski's numeration

Let x be an irrational number in $(0, 1)$. Every real number $y \in [0, 1)$ can be written uniquely in the form

$$y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}|$$

where

- $0 \leq b_i \leq a_i$ for all $i \geq 1$;
- if $a_i = b_i$ for some i, then $b_{i+1} = 0$;
- $a_i \neq b_i$ for infinitely many indices i.

The sequence of digits $(b_i)_i$ is given by the Ostrowski map applied to (x, y).

Around Ostrowski's numeration

There are

- two dual algorithms for real numbers

$$y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}| \quad \text{vs.} \quad y = \sum_{i=1}^{\infty} b_i \theta_{i-1}$$

- two dual algorithms for integers (cf. Zeckendorf numeration for the Fibonacci base)

$$N = \sum_{i=1}^{\infty} b_i (-1)^{i-1} q_{i-1} \quad \text{vs.} \quad N = \sum_{i=1}^{\infty} b_i q_{i-1}$$

Ostrowski numeration is used e.g. for discrepancy estimates or for inhomomeneous approximation

Inhomogeneous approximation

Ostrowski's numeration is used to

approximate y mod. 1 by numbers of the form kx, $k \in \mathbb{N}$.

We look for a sequence of integers $(N_n)_n$ such that

$$N_n x \to y \text{ modulo } 1$$

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Write y as

$$y = \sum_{i=1}^{\infty} b_i |\theta_{i-1}| = \sum_{i=1}^{\infty} b_i (-1)^{i-1} (q_{i-1} x - p_{i-1})$$

Set

$$N_n := \sum_{i=1}^{n} (-1)^{i-1} b_i q_{i-1}$$

$$N_n x = \sum_{i=1}^{n} b_i (-1)^{i-1} q_{i-1} x \equiv \sum_{i=1}^{n} (-1)^{i-1} b_i (q_{i-1} x - p_{i-1}) \text{ mod } 1$$

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Set

$$N_n := \sum_{i=1}^{n} (-1)^{i-1} b_i q_{i-1} \leadsto |y - N_n x| = \left| \sum_{i=n+1}^{\infty} b_i |\theta_{i-1}| \right|$$

The sequence (N_n) yields the sequence of best (left) inhomogeneous approximation of y modulo 1

Measure-theoretic results

For continued fractions

$$\lim \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log 2} \quad \text{a.e. } x$$

For Ostrowski's numeration [Ito-Nakada]

$$\lim \frac{1}{n} \log N_n^* = \frac{\pi^2}{12 \log 2} \quad \text{a.e. } (x, y)$$

with

$$N_n^* x \to y \text{ modulo } 1$$

[Ito] There is an invariant measure μ for the Ostrowski map and the density is explicitly given by

$$\begin{cases} \frac{1}{2 \log 2} \frac{x+3}{(1+x)^2} & \text{when } x \geq y \\ \frac{1}{2 \log 2} \frac{x+2}{(1+x)^2} & \text{when } x < y, \end{cases}$$

obtained by providing an explicit realization of the natural extension

Jacobi-Perron versus Ostrowski

The Jacobi-Perron map is defined on $[0,1]^2$ as

$$(\alpha, \beta) \mapsto (\{\beta/\alpha\}, \{1/\alpha\})$$

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Theorem [Broise-Guivarc'h '99] There exists $\delta > 0$ s.t. for a.e. (α, β), there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}, \ |\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}},$$

where p_n, q_n, r_n are given by the Jacobi-Perron algorithm

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Dynamically, we are between the Gauss map (continued fractions) and the Jacobi–Perron case (dimension 2).

A central limit theorem for Ostrowski's map

We have a central limit theorem for Birkhoff sums

$$S_n(f) = \sum_{i=1}^{n-1} f \circ S^i$$

The ergodic theorem states that for $f \in L_1([0,1]^2)$

$$\frac{1}{n} S_n(f) \to \int f \ d\mu \quad a.e.\ (x, y)$$

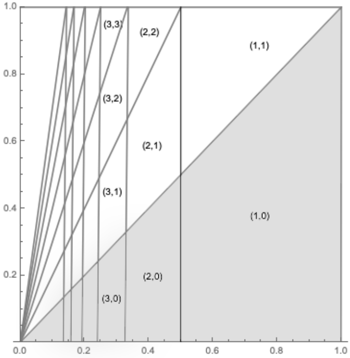
We have more: $S_n f / \sqrt{n}$ converges in law to a normal distribution.

Theorem [B.-Lee] The Ostrowski dynamical system has exponential mixing with respect to its invariant measure μ. For given f with mild assumptions and $z \in \mathbb{R}$

$$\mu \left\{ \frac{1}{\sqrt{n}} S_n f \leq z \right\} \longrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2\sigma^2} dt$$

for a suitable constant σ as n goes to infinity.

On the inverse branches



We consider the partition by fundamental "cylinders" $I_{a,b}$ which corresponds to the (x, y) having as a first pair of digits (a, b). The restriction of the map S on each $I_{a,b}$ is a bijection onto its image $S(I_{a,b})$ given by the homography

$$h_{a,b}(x, y) = \left(\frac{1}{a + x}, \frac{b + y}{a + x} \right).$$

On the Jacobian

Let

$$h_{a,b}(x,y) = \left(\frac{1}{a+x}, \frac{b+y}{a+x} \right).$$

Its Jacobian satisfies

$$\mathbf{J}_{\mathbf{h}_{a,b}}(x,y) = \det \left(\begin{bmatrix} -\frac{1}{(a+x)^2} & 0 \\ -\frac{b+y}{(a+x)^2} & \frac{1}{a+x} \end{bmatrix} \right) = -\frac{1}{(a+x)^3}$$

It is non-vanishing and uniform with regard to the skew coordinate.

More on the fundamental cylinders

Let $\mathcal{A} = \{(a, b) : 0 \leq a \leq b\}$ be the alphabet of digits.
Let $(a, b) = ((a_1, b_1), \cdots, (a_n, b_n)) \in \mathcal{A}^n$. Set

$$q_{-1} = 0,\ p_{-1} = 1,\ q_0 = 1,\ p_0 = 0,$$

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}.$$

Let $h_{a,b} = h_{a_1,b_1} \circ \cdots \circ h_{a_n,b_n}$ be the corresponding inverse branch. Assume $0 \leq b_i < a_i$ for all i. One has

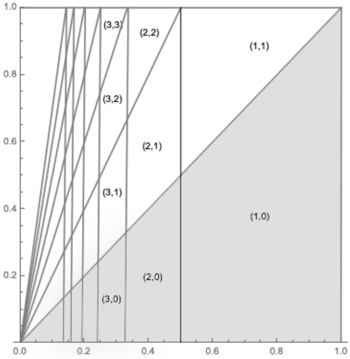
$$I_{a,b} = h_{a,b}([0, 1]^2).$$

The Jacobian determinant $J_{h_{a,b}}$ satisfies

$$|J_{a,b}(x, y)| = \left(\frac{1}{q_n + xq_{n-1}}\right)^3 \text{ for all } (x, y) \in [0, 1]^2$$

$$\rightsquigarrow \text{mes}(I_{a,b}) = 0(1/q_n^3) \text{ and } \text{d}iam(I_{a,b}) = 0(1/q_n).$$

On the Markov partition



Let

$$\Delta_0 = \{(x,y) \in [0,1]^2 : y < x\} \qquad \Delta_1 = \{x,y) \in [0,1]^2 : x \leq y\}$$

$$S(I_{a,b}) = \Delta_0 \text{ if } a = b \qquad S(I_{a,b}) = [0,1]^2 \text{ if } a \neq b$$

$\rightsquigarrow$ if $a_i = b_i$ for some i, then $b_{i+1} = 0$.

Transfer operators

Let f be an observable for a Birkhoff sum. We consider a weighted transfer operator

$$\begin{aligned}\mathcal{L}_{s,w}\phi(x,y) &= \sum_{S(u,v)=(x,y)} |J_S(u,v)|^s e^{wf(u,v)}\phi(u,v) \\ &= \sum_{(a,b)} \frac{\exp\left(w\cdot f\left(\frac{1}{a+x},\frac{b+y}{a+x}\right)\right)}{(a+x)^{3s}}\phi\left(\frac{1}{a+x},\frac{b+y}{a+x}\right)\cdot 1_{SI_{a,b}}(x,y)\end{aligned}$$

The parameter w will be used for the study of probabilistic limit theorems and the parameter s attached to the Jacobian determinant plays a role in the study of Hausdorff dimensions.

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- The goal is to take a suitable functional space on which the transfer operator acts compactly and has a spectral gap.
- $\rightsquigarrow$ There exists an ergodic absolutely continuous S-invariant measure μ so that μ that has exponential mixing properties. Thus we have central limit theorems for Birkhoff sums and we also get a spectral description for $\dim_H(E_N)$

On a suitable functional space

We follow the approach developed by Mayer for locally expanding maps and Broise for Jacobi–Perron algorithm
$\rightsquigarrow$ This gives a functional space on which the operator is compact. In particular, it has a simple largest eigenvalue whose modulus is strictly larger than all other eigenvalues (cf. Perron-Frobenius theorem.)

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A contraction property There exists a domain Ω that contains $[0,1]^2$ and that is mapped by the inverse branches of the dynamical system strictly into itself.

We work with the functional space of $L_1([0,1]^2)$ functions that can be continued as bounded holomorphic functions that are defined on the complexification $\Omega + i\Omega$ and that take into account the Markov property (if $a_i = b_i$ for some i, then $b_{i+1} = 0$)

$$f \equiv 1_{\Delta_0} f_0 + 1_{\Delta_1} f_1 \equiv (f_0, f_1)$$

On the invariant measure

How to get an acim?

- Use a realization of the natural extension $({}^tM^{-1}, M)$ [Arnoux-Labbé]
- Use the transfer operators $\rightsquigarrow$ CLT for Birkhoff sums.

Back to the natural extension

According to Ito-Nakada

$$\overline{S}^n(x,y,0,0) = (S^n(x,y), \frac{q_{n-1}}{q_n}, \frac{N_n^*}{q_n}) \text{ with } N_n^* = \sum_{i=1}^{n} b_i q_{i-1}$$

Theorem [Ito-Nakada] For a.e. (x,y)

$$\lim \frac{1}{n} \log N_n^* = \frac{\pi^2}{12 \log 2}$$

$$\lim_n 1/n \,\mathrm{Card}\{k : q_k|y - \sum_{i=1}^{k} b_i\theta_{i-1}| \leq z,\ 1 \leq k \leq n\} = z(1-z)$$

for $0 \leq z \leq 1$

Continued fractions and Hausdorff dimension

Let

$$E_N = \{x = [0; a_1, a_2, \cdots] : a_i \leq N \text{ for all } i\}$$

A powerful approach to $\dim_H(E_N)$ is provided by the thermodynamic formalism via transfer operators

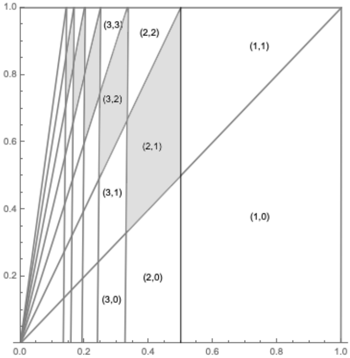
- Consider constrained transfer operators

$$\mathcal{L}_s f(x) = \sum_{a \leq N} \frac{1}{(a+x)^{2s}} f\left(\frac{1}{a+x}\right).$$

- Bowen-Ruelle type formula
  One has $\dim_H(E_N) = s$ if and only if s is the unique solution to $\lambda_s = 1$, where λ_s is the dominant eigenvalue.
- $\rightsquigarrow$ Hensley's formula

$$\dim_H(E_N) = 1 - \frac{6}{\pi^2}\frac{1}{N} - \frac{72}{\pi^4}\frac{\log N}{N2} + O\left(\frac{1}{N^2}\right)$$

Back to Ostrowski's map



We consider the set

$$E_N = \{(x, y) \in [0, 1)^2 : a_i \leq N \text{ and } b_i = a_i - 1, \text{ for all } i \geq 1\}$$

Proposition Consider the restriction $S|_{E_N}$ and its associated transfer operator $\mathcal{L}_{N,s}$. Then we have

$$s_1 \leq \dim_H(E_N) \leq s_2$$

for real variables s_1, s_2 satisfying $\lambda_{N,s_1} = 1$ and $N^2\lambda_{N,s_2} = 1$.

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Proof We use the fact that

$$\mathcal{L}^n_{N,s}\mathbf{1}(x,y) = \sum_{(a,b)\in\mathcal{A}^n_N} |\mathbf{J}_{a,b}(x,y)|^s$$

For the lower bond, we use the probability eigenmeasure $\mu_{N,s}$ of $\lambda_{N,s}$ for the dual operator with the the mass distribution principle:
If for all A, $\mu(A) \le C\mathrm{Diam}\ (A)^t$, then $\dim_H(X) \ge t$.

And now...

- Study of the quadratic case (quadratic parameters are characterized by periodic expansions)
- Study of the integer case
- Other skew products of the Gauss map
- More precise estimates for the Hausdorff dimension
- Numerical estimates
- Toward multidimensional continued fraction algorithms